A Lightweight Derivation of CORDIC for Trigonometric Functions

Abstract

A short, simple derivation of CORDIC using matrices and infinite series, abstracted away from hardware details.

Let

$$M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

It follows, by factoring out a $\cos\theta$ from M,

$$M = \cos\theta \begin{pmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{pmatrix}.$$

Set

$$\theta_0 = 0, \quad \theta_{n+1} = \sum_{k=0}^{n} a_k \tan^{-1}\left(2^{-k}\right),$$

With

$$a_k = \begin{cases} +1 & \text{if } \theta_k < \theta, \\ -1 & \text{if } \theta_k > \theta. \end{cases}$$

That is to say, write θ as an infinite summation of arctangents of the reciprocals of powers of two using the algorithm seen in the definition of a_k, which can be surmised as follows:

> If the current angle approximation *overshoots* the target, subtract, otherwise, add.

Note that, for

$$\tan\alpha = 2^{-k},$$

we have

$$\cos\alpha = \frac{1}{\sqrt{1 + 2^{-2k}}}.$$

Hence,

$$M = \prod_{k=0}^{\infty} \frac{1}{\sqrt{1+2^{-2k}}} \begin{pmatrix} 1 & -a_k 2^{-k} \\ a_k 2^{-k} & 1 \end{pmatrix}.$$

We use the fact that $a_k \in \{-1, 1\}$ as to move it within the value of the tangent. Splitting the product,

$$M = \left(\prod_{k=0}^{\infty} \frac{1}{\sqrt{1+2^{-2k}}} \right) \left(\prod_{k=0}^{\infty} \begin{pmatrix} 1 & -a_k 2^{-k} \\ a_k 2^{-k} & 1 \end{pmatrix} \right).$$

We see that our left product is independent of θ, and can be treated as a constant.

$$\prod_{k=0}^{\infty} \frac{1}{\sqrt{1+2^{-2k}}} \approx 0.607252935\ldots$$

The decimal expansion of this constant is sequence A273413 in the OEIS

We can then write

$$M = (0.607252935\ldots) \prod_{k=0}^{\infty} \begin{pmatrix} 1 & -a_k 2^{-k} \\ a_k 2^{-k} & 1 \end{pmatrix}.$$

Expanding this infinite product into an iterative formula, we get

$$\begin{aligned} x_{k+1} &= x_k - a_k 2^{-k} y_k, \\ y_{k+1} &= y_k + a_k 2^{-k} x_k, \end{aligned} \qquad k \in \mathbb{N} \cup \{0\}$$

$$\lim_{n\to\infty} \begin{pmatrix} 0.60725\ldots \cdot x_n \\ 0.60725\ldots \cdot y_n \end{pmatrix} = M \begin{pmatrix} x \\ y \end{pmatrix}.$$